Problem:

Find the general solution of the second order differential equation with constant coefficients:

$y'' + 9y = 5e^{3x}$.

Solution:

First let's solve the homogeneous equation $y'' + 9y = 0$. Its characteristic equation will be:

$\lambda^2 + 9 = 0 \Rightarrow \lambda_1 = 3i, \lambda_2 = -3i, \Rightarrow$ the general solution of the homogeneous equation will be:

$y_1 = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = C_1 e^{3ix} + C_2 e^{-3ix} = (C_1 + C_2)\cos 3x + i(C_1 - C_2)\sin 3x$, passing to real coefficients, we obtain $\Rightarrow y_1 = C_1 \cdot \cos 3x + C_2 \cdot \sin 3x$.

Let's look for the particular solution of the initial equation in the form $y_2 = ae^{3x} \Rightarrow y_2' = 3ae^{3x}, y_2'' = 9ae^{3x}$

$\Rightarrow y_2'' + 9y_2 = 18ae^{3x} = 5e^{3x} \Rightarrow a = \dfrac{5}{18} \Rightarrow y_2 = \dfrac{5}{18}e^{3x} \Rightarrow$

the general solution of the initial equation will be:

$$\boxed{y = y_1 + y_2 = C_1\cos 3x + C_2\sin 3x + \frac{5}{18}e^{3x}}.$$