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Problem:

Find the normal form of the quadratic form using the Lagrange method:

$f = x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_2x_3.$

Solution:

Using the Lagrange method, let's separate the full squares:

$$f(x_1, x_2, x_3) = x_1^2 + 2x_1 \cdot 2x_2 + 4x_2^2 - 3x_2^2 + 3x_3^2 + 2x_2x_3 =$$

$$= (x_1 + 2x_2)^2 - \left(\left(\sqrt{3}x_2\right)^2 - 2 \cdot \sqrt{3}x_2 \cdot \frac{1}{\sqrt{3}}x_3 + \left(\frac{1}{\sqrt{3}}x_3\right)^2\right) + \frac{1}{3}x_3^2 + 3x_3^2 =$$

$$= (x_1 + 2x_2)^2 - \left(\sqrt{3}x_2 - \frac{1}{\sqrt{3}}x_3\right)^2 + \left(\frac{\sqrt{10}}{\sqrt{3}}x_3\right)^2,$$

⇒ making a replacement of variables

$$\begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = \sqrt{3}x_2 - \frac{1}{\sqrt{3}}x_3 \\ y_3 = \sqrt{\frac{10}{3}}x_3 \end{cases}$$

⇒ we obtain the desired normal form of the original quadratic form:

$\boxed{f = y_1^2 - y_2^2 + y_3^2}.$

(the normal form of the real quadratic form is $f = \sum_{i=1}^{n} \mathcal{E}_i \cdot y_i^2, where\ \mathcal{E}_i = \pm 1, i = \overline{1, n}$)