https://mathproblemsbank.net/
support@mathproblemsbank.net

Problem:

Find out, does the set of all real numbers forms the linear space if the sum of any two elements $a \; and \; b$ is defined in it, equal to $a + b$ and the product of any element a by any real number ε, equal to $\varepsilon \cdot a$.

Solution:

Let's find out, does the set $\mathbb{R}$ forms the linear space with the operations of addition and multiplication by the number defined on it: $< \mathbb{R}, +, \cdot >$.

To do this, let's check the satisfiability of the linear space axioms.

First, let's check the closedness with respect to the indicated operations:

$\forall a, b \in \mathbb{R} \Rightarrow a + b \in \mathbb{R}, \forall a \in \mathbb{R}, \forall \varepsilon \in \mathbb{R} \Rightarrow \varepsilon \cdot a \in \mathbb{R} \Rightarrow$ $\mathbb{R}$ closed under these operations.

a) $\forall a, b \in \mathbb{R} \Rightarrow a + b = b + a$ (standard addition of numbers),

b) $\forall a, b \; c \in \mathbb{R} \Rightarrow (a + b) + c = a + (b + c) = a + b + c$,

c) $0 \in \mathbb{R}, a + 0 = 0 + a = a, \forall a \in \mathbb{R} \Rightarrow 0 -$ neutral element with respect to addition,

d) $\forall a \in \mathbb{R} \Rightarrow -a \in \mathbb{R}, a + (-a) = -a + a = 0, \Rightarrow \forall a \in \mathbb{R}$ there is an inverse element by addition:

$-a$ (opposite element),

e) $\forall a \in \mathbb{R} \Rightarrow \forall \varepsilon, \beta \in \mathbb{R} \Rightarrow \varepsilon(\beta a) = \varepsilon\beta a = (\varepsilon\beta)a$,

f) $1 \cdot a = a \; \forall a \in \mathbb{R}$ (neutral element by multiplication),

g) $\forall \alpha \in \mathbb{R}, \forall \varepsilon, \beta \in \mathbb{R} \Rightarrow (\varepsilon + \beta)a = \varepsilon a + \beta a$,

h) $\forall \varepsilon \in \mathbb{R}, \forall a, b \in \mathbb{R} \Rightarrow \varepsilon(\alpha + b) = \varepsilon a + \varepsilon b$.

As you can see, all linear space axioms are satisfied $\Rightarrow < M, +, \cdot >$ will be a linear space.