Problem:

Show that the gradient field of the scalar field $\varphi = 6x^3y - 2xy^4 + z^4x^2y$ is irrotational.

Solution:

$\varphi = 6x^3y - 2xy^4 + z^4x^2y$.

Show that the gradient field is irrotational means that the rotor of gradient φ must be zero:

$$\operatorname{grad}\varphi = \frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath} + \frac{\partial\varphi}{\partial z}\vec{k}, \qquad P = \frac{\partial\varphi}{\partial x} = 18x^2y - 2y^4 + 2z^4xy, \qquad Q = \frac{\partial\varphi}{\partial y} = 6x^3 - 8xy^3 + z^4x^2,$$

$$R = \frac{\partial\varphi}{\partial z} = 4z^3x^2y, \qquad \operatorname{grad}\varphi = P\vec{\imath} + Q\vec{\jmath} + R\vec{k} \Rightarrow$$

$$\Rightarrow \operatorname{rot}\operatorname{grad}\varphi = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k},$$

but $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = 4z^3x^2, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 8z^3xy, \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 18x^2 - 8y^3 + 2z^4x \Rightarrow \operatorname{rot}\operatorname{grad}\varphi = 0 \Rightarrow$

the gradient field is irrotational.