Problem:

Prove that the set of rational numbers $\mathbb{Q}$ is countable.

Solution:

Let's prove that $\mathbb{Q}$ is countable.

The set of rational numbers $\mathbb{Q} = \left\{\frac{m}{n} \mid n \neq 0, m, n \in \mathbb{Z}\right\}$, we can assume that the denominator of any rational number is positive, and the sign of the number coincides with the sign of the numerator ⇒

$\Rightarrow \mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ for any $m \in \mathbb{Z}$.

Let's denote

$$Q_m = \left\{\frac{m}{n} \mid n \in \mathbb{N}\right\} = \left\{\frac{m}{1}; \frac{m}{2}; \frac{m}{3}; \dots\right\} \Rightarrow \mathbb{Q} = \bigcup_{m \in \mathbb{Z}} Q_m.$$

Now we obviously have a one-to-one correspondence $Q_m \leftrightarrow \mathbb{N}, (\forall m \in \mathbb{Z})$, since $\frac{m}{n} \leftrightarrow n, n = 1,2, \dots$, it means that Q_m is countable when $m \in \mathbb{Z}$. But the set of integers is countable, and we know that the union of countable numbers of countable sets is countable $\Rightarrow \bigcup_{m \in \mathbb{Z}} Q_m$ is also countable, but $\mathbb{Q} = \bigcup_{m \in \mathbb{Z}} Q_m \Rightarrow \mathbb{Q}$ is countable.