Problem:

Calculate the integral using residues:

$$\int_0^{2\pi} \frac{dt}{7+\sqrt{13}\sin t}.$$

Solution:

$$I=\int_0^{2\pi} \frac{dt}{7+\sqrt{13}\sin t}$$

$1/(7+\sqrt{13}\sin t)$ is a rational function of $\sin t$, continuous in $[0; 2\pi]$ $(7+\sqrt{13}\sin t \geq 7-\sqrt{13}>0) \Rightarrow$ let's denote $z=e^{it}, \Rightarrow$

$$\Rightarrow \sin t=\frac{1}{2i}\left(e^{it}-e^{-it}\right)=\frac{1}{2i}\left(z-\frac{1}{z}\right)=\frac{z^2-1}{2iz}, \qquad dz=ie^{it}dt=izdt \Rightarrow dt=\frac{dt}{it}, \qquad |z|=1, \Rightarrow$$

$$\Rightarrow I=\int_0^{2\pi} \frac{dt}{7+\sqrt{13}\sin t}=\oint_{|z|=1} \frac{1}{7+\sqrt{13}\cdot\frac{z^2-1}{2iz}}\cdot\frac{dz}{iz}=\oint_{|z|=1} \frac{2}{\sqrt{13}z^2+14iz-\sqrt{13}}\cdot dz=$$

$$=\frac{2}{\sqrt{13}}\cdot\oint_{|z|=1} \frac{1}{z^2+\frac{14i}{\sqrt{13}}z-1}dz=\frac{2}{\sqrt{13}}\oint_{|z|=1} f(z)\,dt, \qquad f(z)=\frac{1}{z^2+\frac{14i}{\sqrt{13}}z-1}=\frac{1}{(z+\sqrt{13}i)\left(z+\frac{i}{\sqrt{13}}\right)} \Rightarrow$$

$\Rightarrow f(z)$ has two singular points $z_1=-\sqrt{13}i, z_2=-i/\sqrt{13}$, of them only z_2 lies in the disc $|z|\leq 1$

$\left(|z_2|=\frac{1}{\sqrt{13}}<1, |z_1|=\sqrt{13}>1\right) \Rightarrow$ according to the main residue theorem $\Rightarrow$

$$\oint_{|z|=1} f(z)\,dt=2\pi i\cdot\operatorname*{Res}_{z_2} f(z), z_2 \text{ is a 1st order pole} \Rightarrow \operatorname*{Res}_{z_2} f(z)=\lim_{z\to z_2}(z-z_2)f(z)=$$

$$=\lim_{z\to-\frac{i}{\sqrt{13}}} \frac{1}{z+\sqrt{13}i}=-\frac{\sqrt{13}i}{12} \Rightarrow I=\frac{2}{\sqrt{13}}\cdot 2\pi i\cdot\left(-\frac{\sqrt{13}i}{12}\right)=\frac{\pi}{3}.$$

Answer: $\frac{\pi}{3}$.