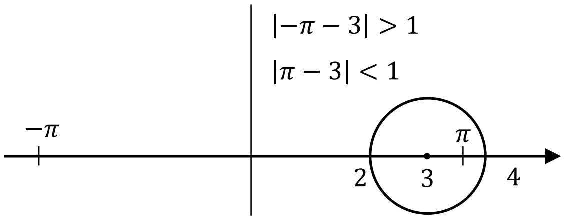
Problem:

Calculate the integral of the complex variable:

$$\oint\limits_{|z-3|=1} \frac{1-\cos z}{z^2-\pi^2}\,dz.$$

Solution:

$$\mathrm{I} = \oint\limits_{|z-3|=1} \frac{1-\cos z}{z^2-\pi^2}\,dz.$$



$$f(z) = \frac{1-\cos z}{z^2-\pi^2} = \frac{1-\cos z}{(z-\pi)(z+\pi)} \Rightarrow$$

$f(z)$ has two singular points $z_1 = \pi, z_2 = -\pi$, of these points only z_1 lies in the circle $|z-3| < 1, \Rightarrow$ from the main residue theorem$\Rightarrow$

$$\Rightarrow I = \oint\limits_{|z-3|=1} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_1} f(z)\,, \text{но } f(z) \underset{z\to\pi}{\longrightarrow} \infty, (z-\pi)f(z) = \frac{1-\cos z}{z+\pi} \underset{z\to\pi}{\longrightarrow} \frac{2}{2\pi} \neq \infty,$$

$\Rightarrow z_1 = \pi$ is a 1st order pole $\Rightarrow$ the residue in it will be:

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to\pi}\big((z-\pi)f(z)\big) = \lim_{z\to\pi}\frac{1-\cos z}{z+\pi} = \frac{2}{2\pi} = \frac{1}{\pi} \Rightarrow I = 2\pi i \cdot \frac{1}{\pi} = 2i.$$

Answer: $2i$.