Problem:

Let $(a, b) = 1$. Prove that $(a + b, a - b) = 1$ or 2.

Solution:

$(a, b) = 1$. Let's prove that $(a + b, a - b) = 1$ or 2.

Let $(a + b, a - b) = d \in \mathbb{N} \Rightarrow (*)\ a + b = xd, a - b = yd$, where $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Let's summarize these equalities, and also subtract:

$$\begin{aligned} a + b + (a - b) = (x + y)d \Rightarrow 2a = (x + y)d \\ a + b - (a - b) = (x - y)d \Rightarrow 2b = (x - y)d \end{aligned} \quad (**)$$

First, from $(*)$ if $(a, d) > 1$ and $(a + b) \vdots d, \Rightarrow (a + b) \vdots (a, d), a \vdots (a, d) \Rightarrow b \vdots (a, d) \Rightarrow a, b \vdots (a, d) > 1 \Rightarrow (a, b) > 1 \Rightarrow$ a contradiction $\Rightarrow (a, d) = 1$. Similarly, $(b, d) = 1$. From $(**) \Rightarrow 2a \vdots d$, but $(a, d) = (b, d) = 1 \Rightarrow 2 \vdots d, d \in \mathbb{N} \Rightarrow d = 1$ or $d = 2$, but $d = (a + b, a - b)$. The assertion has been proved.