Problem:

Find all the values of a, for which the system of inequalities

$$\begin{cases} x^2+(5a+2)x+4a^2+2a<0 \\ x^2+a^2=4 \end{cases}$$

has at least one solution.

Solution:

The discriminant of the 1-st inequality will be:

$$D=(5a+2)^2-4(4a^2+2a)=25a^2+20a+4-16a^2-8a=9a^2+12a+4=(3a+2)^2 \Rightarrow$$

$\Rightarrow$ the roots of the trinomial $x^2+(5a+2)x+4a^2+2a$ will be:

$$x_{1,2}=\frac{-5a-2\pm(3a+2)}{2}\Rightarrow x_1=\frac{-5a-2-3a-2}{2}=-4a-2, x_2=\frac{-5a-2+3a+2}{2}=-a\Rightarrow$$

$$\Rightarrow x^2+(5a+2)x+4a^2+2a=(x-x_1)(x-x_2)=(x+4a+2)(x+a), \Rightarrow \text{from } (x-x_1)(x-x_2)<0\Rightarrow$$

$$\Rightarrow \min\{x_1,x_2\}<x<\max\{x_1,x_2\}, i.e. x_1\neq x_2.$$

Let's consider the cases:

$$a)\ x_1>x_2\Rightarrow -4a-2>-a\Rightarrow a<-\frac{2}{3}, \text{from the } 2-\text{nd equation of the system}\Rightarrow$$

$$\Rightarrow x=\pm\sqrt{4-a^2}, x_2<x<x_1\Rightarrow$$

$$\Rightarrow \frac{2}{3}<-a<x<-4a-2\Rightarrow x>\frac{2}{3}>0\Rightarrow x=\sqrt{4-a^2}\Rightarrow -a<\sqrt{4-a^2}<-4a-2,$$

all parts are positive, let's square $\Rightarrow a^2<4-a^2<16a^2+16a+4\Rightarrow a^2<2,$

$$a(17a+16)>0, \qquad a<-\frac{2}{3}<0\Rightarrow 17a+16<0\Rightarrow a<-\frac{16}{17}<-\frac{2}{3}, \qquad a^2<2\Rightarrow |a|<\sqrt{2}\Rightarrow$$

$$\Rightarrow \boxed{-\sqrt{2}<a<-\frac{16}{17}}.$$

$$b)\ x_1<x_2\Rightarrow -4a-2<-a\Rightarrow a>-\frac{2}{3}.$$

$$If\ a\geq 0\Rightarrow -a\leq 0\Rightarrow x=-\sqrt{4-a^2}, \qquad -4a-2<-\sqrt{4-a^2}<-a, \Rightarrow a<-\sqrt{4-a^2}<4a+2\Rightarrow$$

$$\Rightarrow a^2<2, a(17a+16)>0, \qquad a\geq 0\Rightarrow 17a+16>0, \qquad a^2<2\Rightarrow \boxed{0<a<\sqrt{2}}.$$

$$If\ a<0\Rightarrow -\frac{2}{3}<a<0, -4a-2<x<-a, \ \ if -4a-2>0\Rightarrow a<-\frac{1}{2}\Rightarrow x=\sqrt{4-a^2}\Rightarrow$$

$\Rightarrow -4a-2<\sqrt{4-a^2}<-a$, all parts are positive again, $(4a+2)^2<4-a^2<a^2\Rightarrow a^2>2\Rightarrow$

$\Rightarrow a<-\sqrt{2}, \Rightarrow -\frac{2}{3}<a<-\sqrt{2}\Rightarrow 2>3\sqrt{2},\ \ 4>18$ a contradiction $\Rightarrow\ -4a-2\leq 0\Rightarrow a\geq -\frac{1}{2}.$

$If\ x=\sqrt{4-a^2}\Rightarrow\sqrt{4-a^2}<-a\Rightarrow a^2>2,-\frac{2}{3}<a<-\sqrt{2},$ a contradiction again $\Rightarrow x=-\sqrt{4-a^2},$

$-4a-2<-\sqrt{4-a^2}<-a,\Rightarrow 4a+2>\sqrt{4-a^2}\Rightarrow 16a^2+16a+4>4-a^2\Rightarrow a(17a+16)>0,$

$a<0\Rightarrow 17a+16<0\Rightarrow\ -\frac{2}{3}<a<-\frac{16}{17}\Rightarrow 34>48,$ a contradiction again.

We obtained the following values of a, for which the system has at least one solution:

$\left(-\sqrt{2};-\frac{16}{17}\right)\cup\left(0;\sqrt{2}\right).$

Answer: $\left(-\sqrt{2};-\frac{16}{17}\right)\cup\left(0;\sqrt{2}\right).$