Problem:

Solve the inequality for all non-negative values of the parameter a:

$a^3x^4 + 2a^2x^2 - 8x + a + 4 \geq 0, \qquad a \geq 0.$

Solution:

$a^3x^4 + 2a^2x^2 - 8x + a + 4 = ax^2a^2x^2 + ax^2(a^2x^2 + 2ax) - 2x(a^2x^2 + 2ax) + 4ax^2 + 2a^2x^2 -$

$-8x + a + 4 = (ax^2 - 2x)(a^2x^2 + 2ax) + 4(ax^2 - 2x + 1) + 2a^2x^2 + a = (ax^2 - 2x)(a^2x^2 + 2ax) +$

$+4(ax^2 - 2x + 1) + a(ax^2 - 2x + 1) + 2ax + a^2x^2 = (a^2x^2 + 2ax)(ax^2 - 2x + 1) + 4(ax^2 - 2x + 1) +$

$+a(ax^2 - 2x + 1) = (ax^2 - 2x + 1)(a^2x^2 + 2ax + a + 4) \geq 0.$

Note that the 2nd multiplier $a^2x^2 + 2ax + a + 4 = (ax + 1)^2 + a + 3 > 0$, because $a \geq 0 \Rightarrow$

$\Rightarrow ax^2 - 2x + 1 \geq 0, \qquad \frac{D}{4} = 1 - a, in\ case\ of\ 1 - a < 0 \Rightarrow$

$\Rightarrow the\ solution\ to\ the\ inequality\ will\ be\ the\ number\ line\ x \in (-\infty; +\infty),$

$when\ 1 - a \geq 0 \Rightarrow 0 \leq a \leq 1 \Rightarrow\ in\ case\ of\ a = 0, x \in \left(-\infty; \frac{1}{2}\right],\ and\ in\ case\ of\ 0 < a \leq 1 \Rightarrow$

$\Rightarrow x \in (-\infty; \frac{1 - \sqrt{1 - a}}{a}] \cup [\frac{1 + \sqrt{1 - a}}{a}; +\infty).$

$Answer:\ a = 0, x \in (-\infty; \frac{1}{2}], 0 < a \leq 1, x \in (-\infty; \frac{1 - \sqrt{1 - a}}{a}] \cup [\frac{1 + \sqrt{1 - a}}{a}; +\infty),\ \ a > 1, x \in (-\infty; +\infty).$