Problem:

Solve the equation:

$2\log_6\left(\sqrt{x}+\sqrt[4]{x}\right)=\log_4 x.$

Solution:

Let's denote $\sqrt[4]{x}=t\Rightarrow x=t^4, t>0.\Rightarrow 2\log_6(t^2+t)=\log_4 t^4,$

$\log_4 t^4=4\log_{2^2} t=4\frac{1}{2}\log_2 t=2\log_2 t,\qquad 2\log_6(t^2+t)=2\log_6\big(t(t+1)\big)=2\log_6 t+2\log_6(t+1),$

$2\log_6 t+2\log_6(t+1)=2\log_2 t\Rightarrow \log_6 t+\log_6(t+1)=\log_2 t,$

$\log_6(t+1)=\log_2 t-\log_6 t$, если $t=1\Rightarrow x=1$, *but then* $2\log_6 2=\log_4 1=0$, *which is impossible* $\Rightarrow$

$\Rightarrow x\neq 1, t\neq 1\Rightarrow$ *we can write* $\log_2 t=\frac{1}{\log_t 2}\Rightarrow \log_2 t-\log_6 t=\frac{1}{\log_t 2}-\frac{1}{\log_t(2\cdot 3)}=$

$=\frac{1}{\log_t 2}-\frac{1}{\log_t 2+\log_t 3}=\frac{\log_t 3}{\log_t 2\cdot\log_t 6}\Rightarrow \log_6(t+1)=\frac{\log_t 3}{\log_t 2\cdot\log_t 6}\Rightarrow\frac{\log_6(t+1)}{\log_6 t}=\frac{\log_t 3}{\log_t 2},$

from the properties of logarithm $\Rightarrow \log_t(t+1)=\log_2 3\ (*).$

Let's consider the function $f(t)=\log_t(t+1)=\frac{\ln(t+1)}{\ln t}, f'(t)=\left(\frac{\ln(t+1)}{\ln t}\right)'=\frac{\frac{\ln t}{t+1}-\frac{\ln(t+1)}{t}}{\ln^2 t}=$

$=\frac{t\ln t-(t+1)\ln(t+1)}{t(t+1)\ln^2 t}=\frac{1}{t(t+1)\ln^2 t}\ln\frac{t^t}{(t+1)^{t+1}}<0$ *when all* $t>0$, *since* $t^t<(t+1)^{t+1}\Rightarrow$

$\Rightarrow f'(t)<0, t\in(0;+\infty)\Rightarrow f(t)$ strictly decreasing $\Rightarrow$ it takes each value at only one point, note that from $(*)$ when $t=2\Rightarrow f(t)=\log_2 3\Rightarrow t=2$ the unique solution of $(*)\Rightarrow\sqrt[4]{x}=2\Rightarrow$

$\Rightarrow x=2^4=16.$

Answer: $x=16.$