Problem:

Extend function $f(x) = x^3$ into Fourier series in cosines on the segment line $[0; 1]$.

Solution:

In order to extend function $f(x)$ into Fourier series on the segment line $[0; l]$ in cosines, function $f(x)$ must be continued on $[-l; 0]$ evenly, since for an even function on $[-l; l]$ the expansion will be in sines:

Fourier series: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$,

where $b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx = 0, \left(\int_{-l}^{l} g(x)\, dx = 0, \text{where } g(x) = f(x) \sin\frac{n\pi x}{l} \text{ is an odd function} \right)$,

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx, \qquad a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx.$$

$f(x) = x^3$ on $[0; 1] \Rightarrow$ is continued on $[-1; 0]$:

$$f(x) = \begin{cases} x^3, & x \in [0; 1] \\ -x^3, & x \in [-1; 0] \end{cases}, \qquad l = 1, \qquad b_n = 0, \qquad a_0 = \int_{-1}^{1} f(x)\, dx = 2 \int_0^1 x^3\, dx = 2 \frac{x^4}{4} \bigg|_0^1 = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = 2 \int_0^1 x^3 \cos(n\pi x)\, dx = \frac{2}{n\pi} \int_0^1 x^3\, d \sin(n\pi x) = \boxed{\text{let's integrate by parts} \Rightarrow} =$$

$$= \frac{2}{n\pi} \left(x^3 \sin(n\pi x) \bigg|_0^1 - \int_0^1 \sin(n\pi x) dx^3 \right) = \boxed{\begin{array}{c} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array} \Rightarrow} = \frac{6}{n^2\pi^2} \int_0^1 x^2\, d \cos(n\pi x) =$$

$$= \frac{6}{n^2\pi^2} \left(x^2 \cos(n\pi x) \bigg|_0^1 - \int_0^1 \cos(n\pi x)\, dx^2 \right) = \frac{6}{n^2\pi^2} \left(\cos n\pi - \frac{2}{n\pi} \int_0^1 x d \sin(n\pi x) \right) =$$

$$= \frac{6}{n^2\pi^2} \left(\cos n\pi - \frac{2}{n\pi} \left(x \sin(n\pi x) \bigg|_0^1 - \int_0^1 \sin(n\pi x)\, dx \right) \right) = \frac{6}{n^2\pi^2} \left(\cos n\pi - \frac{2}{n\pi} \frac{\cos(n\pi x)}{n\pi} \bigg|_0^1 \right) =$$

$$= \frac{6}{n^2\pi^2} \left(\cos n\pi - \frac{2(\cos n\pi - 1)}{n^2\pi^2} \right) = \frac{6}{n^4\pi^4} \left((n^2\pi^2 - 2)(-1)^n + 2 \right).$$

$a_n = \frac{6}{n^4\pi^4} \left((-1)^n (n^2\pi^2 - 1) + 2 \right) \Rightarrow$ the desired Fourier series will be:

$$\boxed{f(x) \sim \frac{1}{4} + \frac{6}{\pi^4} \sum_{n=1}^{\infty} \frac{6 \cdot ((-1)^n (n^2\pi^2 - 1) + 2)}{n^4} \cos(n\pi x)}$$